Prof. Amador Martin-Pizarro — Albert-Ludwigs-Universität Freiburg
Exercises: Charlotte Amann-Bartnick — Winter term 2025/2026

Model Theory
Sheet 5
Deadline: 20.11.25, 2:30 pm.

Exercise 1 (8 points)**.**

Denote by $2^{<\omega}$, resp. 2^{ω}, the set of all finite, resp. infinite, sequences of 0's and 1's. We denote the concatenation of s and t in $2^{<\omega}$ by $s \frown t$. Consider a language $\mathcal{L} = \{P_s \mid s \in 2^{<\omega}\}$ consisting solely of unary predicates P_s for each such sequence s in $2^{<\omega}$. Let $\mathcal{K}$ be the class of $\mathcal{L}$-structures $\mathcal{A}$ whose universe is $P_{\emptyset}^{\mathcal{A}}$ and such that, for s in $2^{<\omega}$, the set $P_s^{\mathcal{A}}$ is the disjoint union of the subsets $P_{s\frown 0}^{\mathcal{A}}$ and $P_{s\frown 1}^{\mathcal{A}}$, both of which are non-empty.

a) Give an axiomatization T of the class $\mathcal{K}$ and show that T is consistent.

   Hint: There is a suitable uncountable model whose elements are functions.

b) Given a model $\mathcal{A}$ of T and an element f of 2^{ω}, show that there exists an elementary extension $\mathcal{A}'$ of $\mathcal{A}$ such that $\bigcap_{n\in\omega} P_{f\restriction_n}^{\mathcal{A}'}$ contains infinitely many elements, where $f_{\restriction_n}$ is the restriction of f to the first n many entries.

c) Using b), show that T is complete and has quantifier elimination.

d) Describe (informally) all types in $S_1(T)$ and show that no type in $S_1(T)$ is isolated.

e) Conclude from e) that T cannot have a prime model.

Exercise 2 (4 points)**.**

Consider the theory DLO of dense linear orders without endpoints in the language $\mathcal{L} = \{<\}$. Recall that DLO is complete and with quantifier elimination. Fix some natural number $n \geq 1$.

a) Show that every type in $S_n(\mathrm{DLO})$ is isolated.

Let now T be an arbitrary complete theory such that every type in $S_n(T)$ is isolated.

b) Show that $S_n(T)$ must be finite.

Exercise 3 (3 points)**.**

Recall that a subset Y of a topological space X is *dense* if $Y \cap U \neq \emptyset$ for every non-empty open set U in X.

Now consider a countable consistent theory T in the language $\mathcal{L}$. Show that the type space $S_n(T)$ has the *Baire property*: for any countable family $(U_m)_{m\in\mathbb{N}}$ of open dense subsets of $S_n(T)$, the intersection $\bigcap_{m\in\mathbb{N}} U_m$ is dense (and thus non-empty).

HINT: Compactness (**surprising!**) and 0-dimensionality.

(Please turn the page!)

The exercise sheets can be handed in in pairs. Submit them in the mailbox 3.19 in the basement of the Mathematical Institute.

Exercise 4 (5 points).

Suppose there is a partial elementary map $f : C \twoheadrightarrow D$ between two substructures $\mathcal{C}$ of $\mathcal{A}$ and $\mathcal{D}$ of $\mathcal{B}$ in the language $\mathcal{L}$. Fix some $n \geq 1$ in $\mathbb{N}$.

The map f induces a map as follows:

$$\begin{aligned} F \quad : S_n^{\mathcal{A}}(C) \quad &\to \quad S_n^{\mathcal{B}}(D) \\ p(\bar{x}) \quad &\mapsto \quad \{\varphi[x, f(\bar{c})] \mid \varphi[\bar{x}, \bar{c}] \ \ C\text{-instance contained in } p\} \end{aligned}$$

a) Show that the map F is well-defined.

b) Show that F is bijective.

c) Show that F is continuous. Conclude that F is a homeomorphism of topological spaces.